Topological groups and spaces $C(X)$ with ordered bases

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An index set $\Sigma \subseteq \mathbb{N}^\mathbb{N}$ is boundedly complete if each bounded subset of $\Sigma$ has an upper bound at $\Sigma$. If $\Sigma$ is unbounded and directed (and if additionally $\Sigma$ is boundedly complete) a base $\{U_\alpha : \alpha \in \Sigma\}$ of neighborhoods of the identity of a topological group $G$ with $U_\beta \subseteq U_\alpha$, whenever $\alpha \leq \beta$ with $\alpha, \beta \in \Sigma$, is called in [2] a $\Sigma$-base (a long $\Sigma$-base). The case $\Sigma = \mathbb{N}^{\mathbb{N}}$ has been noticed for topological vector spaces under the name of $\mathfrak{S}$-base at [1]. If $X$ is a separable, metrizable and not Polish space, the space $C_c(X)$ has a $\Sigma$-base but does not admit any $\mathfrak{S}$-base ([2]). Under an appropriate ZFC model the space $C_c(\omega_1)$ has a long $\Sigma$-base which is not a $\mathfrak{S}$-base ([2]).

In [2] we proved that (i) if $G$ is a topological group with a long $\Sigma$-base then every compact subset of $G$ is metrizable and (ii) that a Fréchet-Urysohn topological group is metrizable if and only if it has a long $\Sigma$-base. This result improve the recent result in [3] stating that a Fréchet-Urysohn topological group with $\mathfrak{S}$-base is metrizable.

By (i) if $C_c(X)$ has a long $\Sigma$-base then every compact subset of $C_c(X)$ is metrizable (i.e., $C_c(X)$ is strictly angelic). Then $X$ is a $C$-Suslin space, and we get that $C_p(X)$ is angelic by Orihuela’s theorem at [4], whence $C_c(X)$ is also angelic. Also we show in [2] that a $C_p(X)$ space has a long $\Sigma$-base if and only if $X$ is countable.

**Problem** We do not know whether there exists a topological group with a long $\Sigma$-base that admits no $\mathfrak{S}$-base.

**Problem** Let $X$ be a separable metric space admitting a compact ordered covering of $X$ indexed by an unbounded and boundedly complete proper subset of $\mathbb{N}^{\mathbb{N}}$ that swallows the compact subsets of $X$. Is then $X$ a Polish space?

References


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