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A Hardy inequality for ultraspherical expansions with an application to the sphere

Alberto Arenas, **Óscar Ciaurri**, **Edgar Labarga**

The Hardy inequality for fractional powers of the classical Laplacian is

$$C_{\sigma,d}\int_{\mathbb{R}^d}\frac{u^2(x)}{|x|^{2\sigma}}\,dx\leq\int_{\mathbb{R}^d}u(x)(-\Delta u(x))^{\sigma}\,dx,$$

where the sharp constant is well-known (cf. [1, 2]).

Consider the family $\{c_n^\lambda\}_{n\geq 0}$ of orthonormal ultraspherical polynomials c_n^λ on $L^2((-1,1),d\mu_\lambda)$, $d\mu_\lambda(x)=(1-x^2)^{\lambda-1/2}\,dx$. Those polynomials are eigenfunctions of the operator $\mathcal{L}_\lambda=(1-x^2)\frac{d^2}{dx^2}-(2\lambda+1)x\frac{d}{dx}-\lambda^2$; that is, $\mathcal{L}_\lambda c_n^\lambda=-(n+\lambda)^2c_n^\lambda$.

Now define the operator

$$A_\sigma^\lambda=\frac{\Gamma\left(\sqrt{-\mathcal{L}_\lambda}+\frac{1+\sigma}{2}\right)}{\Gamma\left(\sqrt{-\mathcal{L}_\lambda}+\frac{1-\sigma}{2}\right)}.$$

In this talk we present a Hardy inequality for the operator A_σ^λ of the form

$$Q_{\sigma,\lambda}\int_{-1}^{1}\frac{u^2(x)}{(1-x^2)^{\sigma/2}}\,d\mu_\lambda(x)\leq\int_{-1}^{1}u(x)A_\sigma^\lambda u(x)\,d\mu_\lambda(x),$$

where the constant $Q_{\sigma,\lambda}=2^\sigma\frac{\Gamma(\lambda/2+(1+\sigma)/4)^2}{\Gamma(\lambda/2+(1-\sigma)/4)^2}$ is sharp.

As a result of this inequality we obtain a Hardy inequality on the sphere involved a potential having singularities in both poles of the sphere that takes the form

$$2^\sigma Q_{\sigma,(d-1)/2}\int_{\mathbb{S}^d}\frac{f^2(\xi)}{(|\xi-e_d||\xi+e_d|)^\sigma}\,d\xi\leq\int_{\mathbb{S}^d}f(\xi)\mathbf{A}_\sigma f(\xi)\,d\xi,$$

with $\mathbf{A}_\sigma=\frac{\Gamma\left(\sqrt{-\Delta_{\mathbb{S}^d}}+\frac{1+\sigma}{2}\right)}{\Gamma\left(\sqrt{-\Delta_{\mathbb{S}^d}}+\frac{1-\sigma}{2}\right)}$ and $-\Delta_{\mathbb{S}^d}$ being the conformal Laplacian on the sphere.

References

[1] W. Beckner, Pitt's inequality and the fractional Laplacian: sharp error estimates, *Forum Math.* **24** (2012), 177–209.

[2] D. Yafaev, Sharp constants in the Hardy-Rellich inequalities, *J. Funct. Anal.* **168** (1999), 121–144.

[1]Departamento de Matemáticas y Computación, Universidad de La Rioja,
Calle Luis de Ulloa s/n, 26004 Logroño, Spain
{alarenas,oscar.ciaurri,edlabarg}@unirioja.es